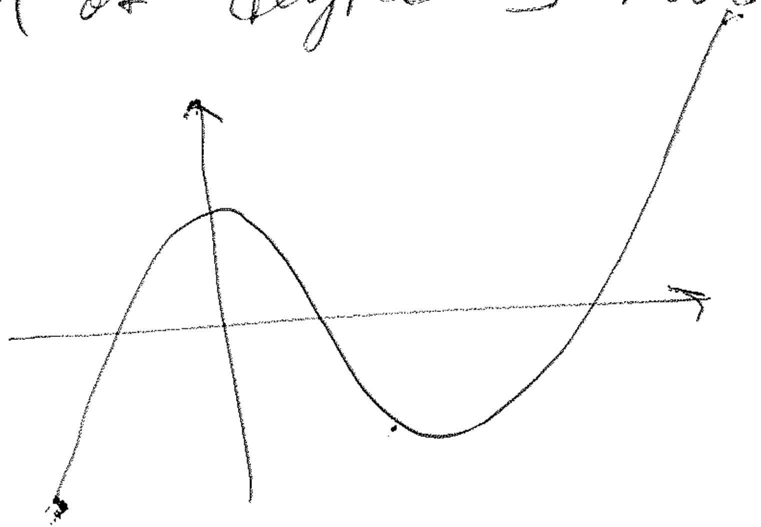
11/04/14. 1

Every polynomial of degree 3 has a real root.



Proof.

Consider

$p(x) = ax^3 + bx^2 + cx + d$,

where $a \neq 0$.

We rewrite

$$p(x) = ax^3\left(1 + \frac{b}{ax} + \frac{c}{ax^2} + \frac{d}{ax^3}\right).$$

(for $x \neq 0$).

$$\lim_{x\to\infty} \left(\frac{b}{ax} + \frac{c}{ax^2} + \frac{d}{ax^3}\right) = 0.$$

Then there is $M > 0$ s.t.

if $x \geq M$, then $\left|\frac{b}{ax} + \frac{c}{ax^2} + \frac{d}{ax^3}\right| < \frac{1}{2}$.

Then $1 + \left(\frac{b}{ax} + \dots\right) > \frac{1}{2}$ and

$$|ax^3|\left(\left(1 + \frac{b}{ax} + \dots\right)\right) > \left|\frac{ax^3}{2}\right|.$$

Now, take $x > M$, then

$$ax^3\left(1 + \frac{b}{ax} + \frac{c}{ax^2} + \frac{d}{ax^3}\right) > \frac{ax^3}{2} > 0.$$

Similarly, there is $\tilde{M} < 0$ s.t. if $x < \tilde{M}$, then

$$ax^3\left(1 + \frac{b}{ax} + \frac{c}{ax^2} + \frac{d}{ax^3}\right) < 0.$$

I.V.T. now implies there exists $x \in \mathbb{R}$ s.t. $p(x) = 0$. $\square$

I'm away — no consult. hours.

Midterm — come early; returned when I'm back

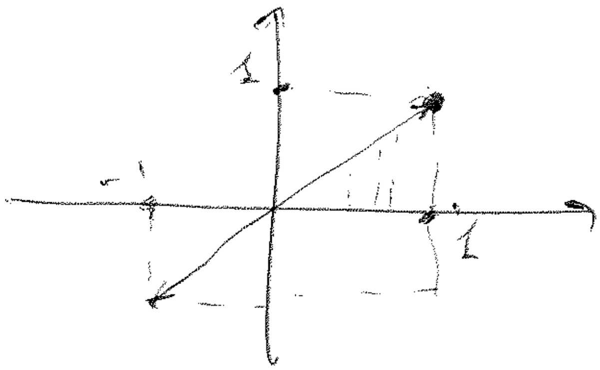
Lec 19.

<u>Def.</u> The function $f : \Omega \to \mathbb{R}$ attains its maximum (minimum) value on Ω at $a \in \Omega$ if $f(a) \geq f(x)$ $(f(a) \leq f(x))$ for all $x \in \Omega$.

The <u>extremal</u> value of f is its max or min value.

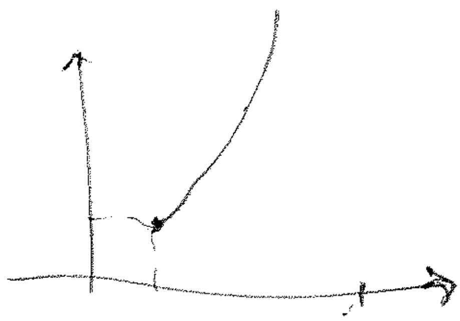
Examples. 1) $f: (-1, 1) \to \mathbb{R}$,

$f(x) = x$.



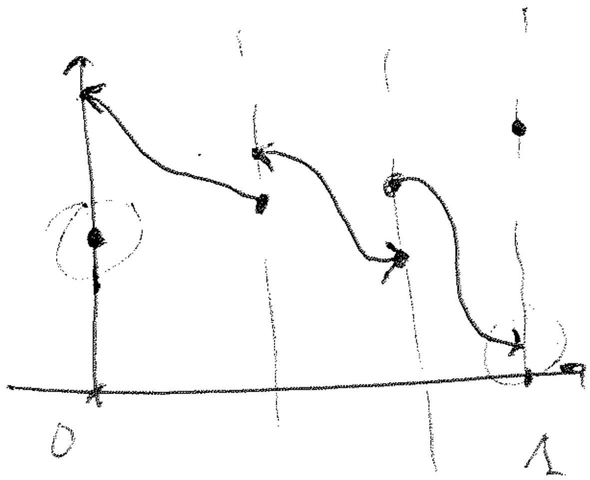
No max, min value.

2) $f(x) = x^2$ on $[1, \infty)$.



min. value at 1.
max value? NO.

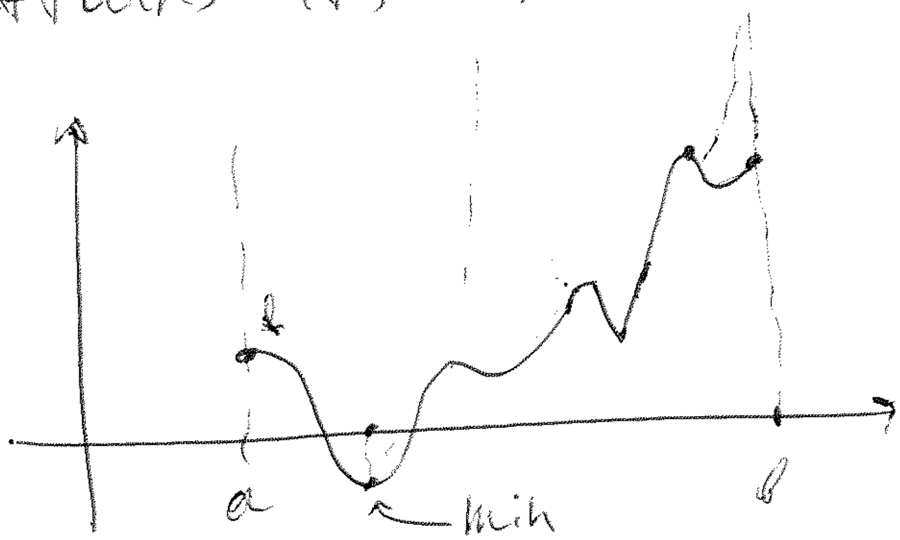
3)



f on $[0, 1]$.

The extreme value theorem (EVT).

Theorem (EVT). Assume f is continuous on $[a,b]$. Then f attains its max and min. value.



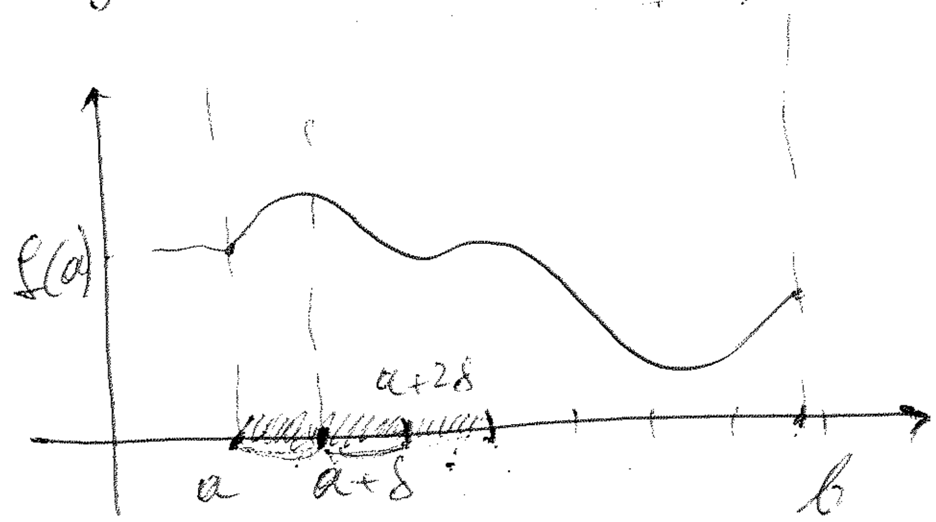
We need a lemma.

Lemma. If f is a contin. function on $[a,b]$, then f is bounded.

Proof. f is contin., therefore, f is uniformly continuous on $[a,b]$. ~~For~~ Take $\varepsilon = 1$ in the def. of uniform continuity. There exists $\delta > 0$ s.t. if $x, y \in [a,b]$ and $|x-y| < \delta$, then $|f(x) - f(y)| < 1$. Take $x = a$. If $|y-a| \leq \delta$, then $|f(y) - f(a)| < 1$.

In other words, if $y \in [a, a+\delta]$, then $|f(a) - f(y)| < 1$ and

$|f(y)| < f(a) + 1$.



Now take $x = a + \delta$.

If $|x - y| < \delta$ (i.e., $|a + \delta - y| \leq \delta$ or $y \in [a, a+2\delta]$), then

$|f(x) - f(y)| \leq 1$ and

$|f(a+\delta) - f(y)| < 1$ or

$$|f(y)| < |f(a+\delta)| + 1 < |f(a)| + 1 + 1 = |f(a)| + 2.$$

Thus, $|f(y)| < |f(a)| + 2$ on $[a, a+2\delta]$.

Now take $x = a + 2\delta$. Continue this procedure. I conclude:

$f(y) \le |f(a)| + n$ on $[a, a+n\delta]$.

We do this n times, where

$n \geq \frac{b-a}{\delta}$. Conclude:

$|f(y)| \le |f(a)| + n$ for $y \in [a,b]$. $\square$

Proof of EVT.

It's enough to do max. To get min, just consider $-f$ and look at its max.

Define $\Theta = \{ f(x) \mid x \in [a,b] \}$. Lemma implies Θ is bounded. Therefore, Θ has a supremum. Call this supremum M.

There exists x_1 s.t.

$M - f(x_1) < 1$; there exists x_2 s.t.

$M - f(x_2) < \frac{1}{2}$; ...; there is x_n s.t.

$M - f(x_n) < \frac{1}{n}$. We get a sequence

$x_1, x_2, \ldots, x_n, \ldots$. This seq. lies in $[a,b]$.

Therefore, $(x_n)_{n=1}^{\infty}$ must have a subseq $(x_{n_k})_{k=1}^{\infty}$ with

$$\lim_{k\to\infty} x_{n_k} = x^* \in [a,b].$$

We know $f(x_{n_k}) > M - \frac{1}{n_k}$.

Therefore,

$\cancel{\lim f(x^*)}$

$$\lim_{k\to\infty} f(x_{n_k}) = f(x^*) \text{ and}$$

$$\lim_{k\to\infty} f(x_{n_k}) \geq M.$$

But $f(x_{n_k}) \leq M$ for all k, so squeeze implies

$$\lim_{k\to\infty} f(x_{n_k}) = M = f(x^*).$$

Thus, f attains max at x^*. $\square$

Ex. 1) $\lim_{x\to 2} (x^2+x) = 6.$ $\boxed{\cancel{7}9/04/14}$

2) $\lim_{x\to a} (x^2+x) = a^2+a.$

Proof. Fix $\varepsilon > 0$. Find $\delta > 0$ s.t.

$0 < |x-a| < \delta \Rightarrow |x^2+x-a^2-a| < \varepsilon.$

Observe that

$$|x^2+x-a^2-a| = |(x^2-a^2)+(x-a)|$$
$$= |(x^{\cancel{2}}-a)(x+a)+(x-a)|$$
$$= \underbrace{|x-a|}\,\underbrace{|x+a+1|}.$$

Notice: if $|x-a| < 1$, then

$|x| < 1+|a|$. Also,

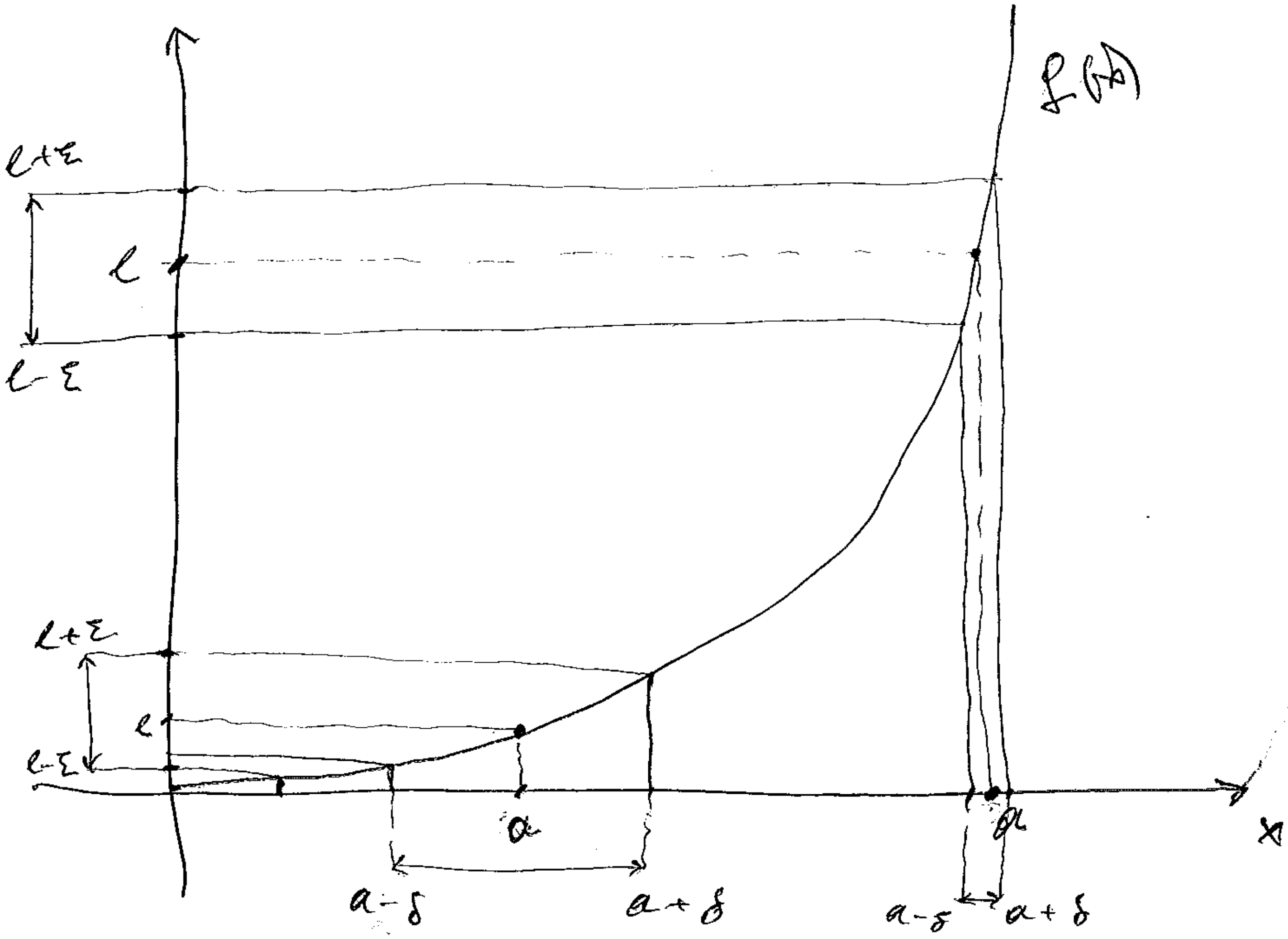
$|(x+a)+1| \le |x|+|a+1| < 1+|a|+|a+1|.$

This means

$$|x^2+x-a^2-a| = |x-a||x+a+1|$$
$$< |x-a|(1+|a|+|a+1|).$$

Take $\delta = \min\left\{\frac{\varepsilon}{1+|a|+|a+1|}, 1\right\}.$ $\square$

Note. δ may depend on ε and a.



Theorem. Suppose $\lim_{x\to a} f(x) = l$, $\lim_{x\to a} g(x) = m$.

1) $\lim_{x\to a}(f(x)+g(x)) = l+m$.

2) $\lim_{x\to a} f(x)g(x) = lm$.

3) If $m \neq 0$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{l}{m}$.

$\boxed{1), 3)}$ - Exercise.

Proof. 2) Take $\varepsilon > 0$. Want $\delta > 0$ s.t. $0 < |x-a| < \delta \Rightarrow |f(x)g(x) - lm| < \varepsilon$.

$$|f(x)g(x) - lm| = |f(x)g(x) + f(x)m - f(x)m - lm|$$

$\leq |f(x)||g(x)-m| + |m||f(x)-\ell|$.

For $\varepsilon = 1$, there is $\delta_1 > 0$ s.t.
$0 < |x-a| < \delta_1 \Rightarrow |f(x)-\ell| < 1$.
Then $|f(x)| < |\ell|+1$.

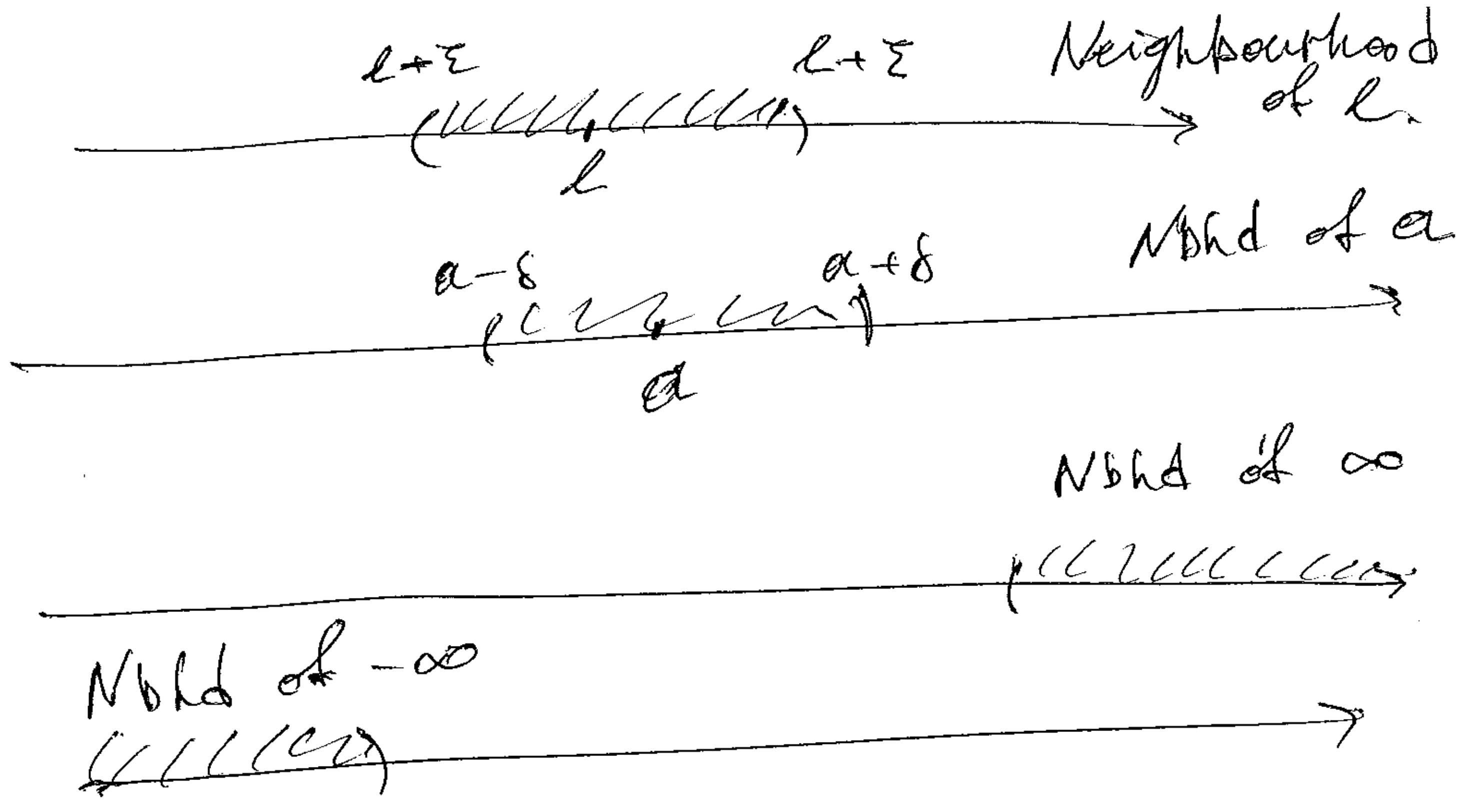
For such x, we have
$$|f(x)g(x)-\ell m| \leq (|\ell|+1)|g(x)-m| + |m||f(x)-\ell|.$$

For $\frac{\varepsilon}{2(|\ell|+1)}$, there exists
$\delta_2 > 0$ s.t. $0 < |x-a| < \delta_2$
$\Rightarrow$ ~~$|f(x)-\ell|$~~ $|g(x)-m| < \frac{\varepsilon}{2(|\ell|+1)}$.

Also, there is $\delta_3 > 0$ s.t.
$0 < |x-a| < \delta_3 \Rightarrow |f(x)-\ell| < \frac{\varepsilon}{2(|m|+1)}$.

Now, take $\delta = \min\{\delta_1, \delta_2, \delta_3\}$.
If $0 < |x-a| < \delta$, then
$$|f(x)g(x)-\ell m| < \cancel{(|\ell|+1)}\frac{\varepsilon}{2\cancel{(|\ell|+1)}} + \frac{|m|\,\varepsilon}{2(|m|+1)} < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon. \quad \square$$

Limits at ∞.



<u>Def.</u> $\lim\limits_{x\to\infty} f(x) = l$ if for every $\varepsilon>0$, there exist M s.t. $x>M \Rightarrow |f(x)-l|<\varepsilon$.

Similarly, define $\lim\limits_{x\to-\infty} f(x)$.

<u>Example.</u> Prove that

$$\lim_{x\to\infty} \frac{x^2+x}{x^2+4x+7} = 1.$$

Take $\varepsilon>0$. Want M s.t. when $x>M$,

$$\left|\frac{x^2+x}{x^2+4x+7} - 1\right| < \varepsilon.$$

$$\left|\frac{x^2+x}{x^2+4x+7} - 1\right| = \left|\frac{\cancel{x^2}+x-\cancel{x^2}-4x-7}{x^2+4x+7}\right|$$

$$= \left| \frac{-3x-7}{x^2+4x+7} \right|. \quad \text{Assume } x > 0.$$

Then this equals

$$\frac{3x+7}{x^2+4x+7} \le \frac{3x+7}{x^2}.$$

Now, assume $x > 3$, then $3x > 7$

and

$$\frac{3x+7}{x^2} \le \frac{3x+3x}{x^2} = \frac{6x}{x^2} = \boxed{\frac{6}{x}}.$$

If $x > \frac{6}{\varepsilon}$, then $\frac{6}{x} < \varepsilon$.

Thus, if $x > M = \max\{0, 3, \frac{6}{\varepsilon}\}$,

then $\left| \frac{x^2+x}{x^2+4x+7} - 1 \right| < \varepsilon$.

Actually, $M = \max\{3, \frac{6}{\varepsilon}\}$. $\square$

Example.

$$\lim_{x \to \infty} \frac{x^2}{x^2 - 4x} = 1.$$

For every $\varepsilon > 0$, there exists $M \in \mathbb{R}$ s.t. $x \geq M \Rightarrow \left| \frac{x^2}{x^2-4x} - 1 \right| < \varepsilon$.

Fix $\varepsilon > 0$. We want to find M.

Observe that

$$\left| \frac{x^2}{x^2-4x} - 1 \right| = \left| \frac{\cancel{x^2} - \cancel{x^2} + 4x}{x^2 - 4x} \right|$$

$$= \left| \frac{4\cancel{x}}{\cancel{x}(x-4)} \right| \quad [\text{assume } x > 4].$$

$$= \frac{4}{x-4}.$$

Now, $\frac{4}{x-4} < \varepsilon$ when

$x - 4 > \frac{4}{\varepsilon}$ or $x > \frac{4}{\varepsilon} + 4$.

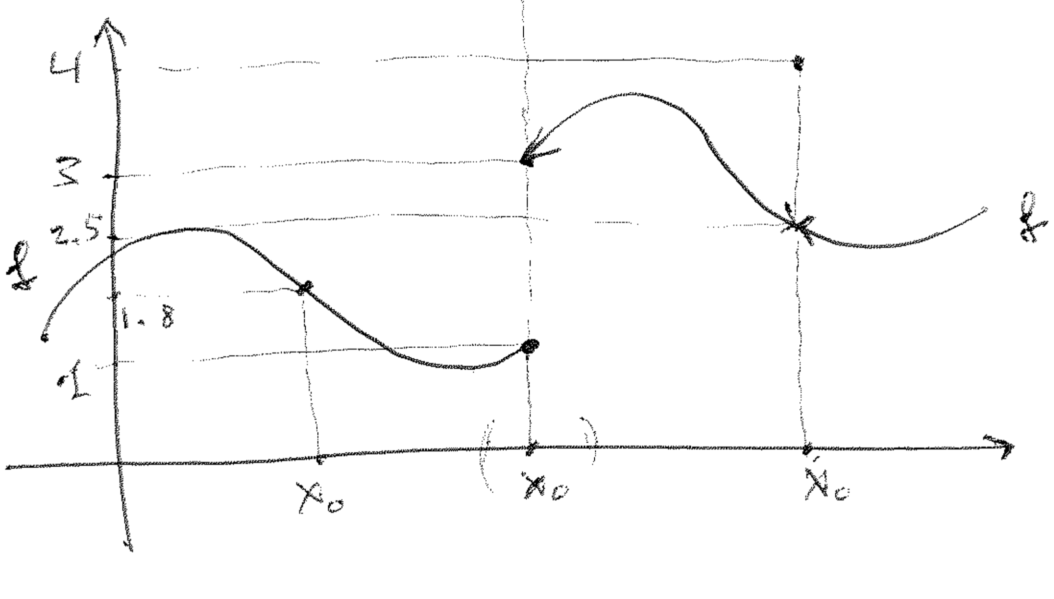
Set $M = \frac{4}{\varepsilon} + 4$. If $x \geq M$, then $\left| \frac{x^2}{x^2-4x} - 1 \right| < \varepsilon$. $\square$

Theorem (Squeeze). Suppose $f, g, h : X \to \mathbb{R}$.
If $f(x) \leq h(x) \leq g(x)$ for all $x \in X$,
and if $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$ for some
a (limit point of X), then
$$\lim_{x \to a} h(x) = \lim_{x \to a} g(x) = \lim_{x \to a} f(x).$$

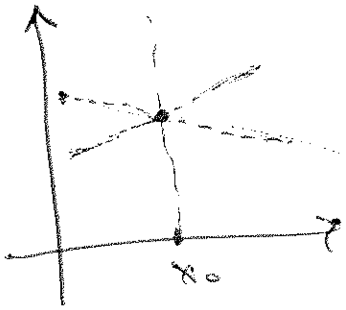
Proof. Too easy. $\square$

may be $\pm\infty$

Def. The function $f : (a, b) \to \mathbb{R}$
is <u>continuous</u> at $x_0 \in (a, b)$ if
$$\lim_{x \to x_0} f(x) = f(x_0), \text{ i.e.,}$$
for every $\varepsilon > 0$, there exists $\delta > 0$
s.t. $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$.



__Note.__ f is continuous on (a,b) if it is continuous at every $x_0 \in (a,b)$.



__Examples.__ 1) $f(x) = x$ is continuous on $\mathbb{R}$.

2) $f(x) = a = \text{const}$ is contin. on $\mathbb{R}$.

__Theorem.__ $f, g : (a,b) \to \mathbb{R}$ are continuous ~~at~~ at $x_0 \in (a,b)$. Then

1) $f+g$ is contin. at x_0

2) fg is contin. at x_0

3) $\frac{f}{g}$ is contin. at x_0 if $g(x_0) \neq 0$.

__Proof.__ Straightforward from the corresponding properties of limits. $\square$

__Corollary.__ Any polynomial or rational function is continuous where defined.

ratio of two polynomials $\left(\text{e.g.}, \frac{x^2+3x+5}{x^7+x+11}\right)$

Example. $f(x)=\frac{1}{x}$ is continuous on $(0,\infty]$.

Prove this using ε-δ definition.

Take $x_0 \in (0,\infty)$. Fix ε.

Want to show: there is $\delta > 0$ s.t.

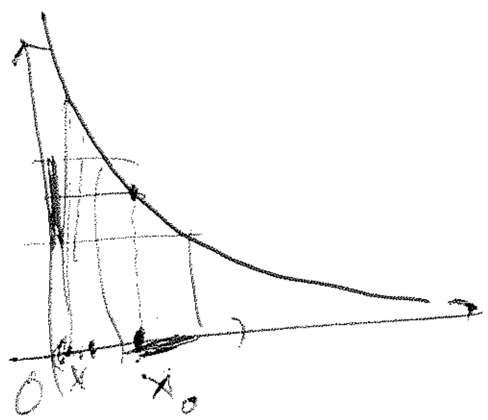
$$|x-x_0|<\delta \Rightarrow \left|\frac{1}{x}-\frac{1}{x_0}\right|<\varepsilon.$$

We compute

$$\left|\frac{1}{x}-\frac{1}{x_0}\right| = \left|\frac{x_0-x}{x\,x_0}\right| = \frac{|x-x_0|}{|x||x_0|}. \qquad \varepsilon\cdot|x||x_0|$$

Need to get rid of the x-dependence downstairs

Assume $\delta < \frac{|x_0|}{2}$.



Then

$$|x-x_0|<\delta \Rightarrow |x-x_0|<\frac{|x_0|}{2},$$

$\cancel{|x|-|x_0|<\frac{|x_0|}{2}}$

$$-\frac{x_0}{2} < x-x_0 < \frac{x_0}{2},$$

$$x-x_0 > -\frac{x_0}{2}, \quad \underline{x > \frac{x_0}{2}}. \text{ Then}$$

$$\left|\frac{x-x_0}{x\,x_0}\right| = \frac{|x-x_0|}{|x|\,x_0} \le \frac{|x-x_0|\,2}{x_0\,x_0} = \frac{2}{x_0^2}|x-x_0|.$$

Take $\delta = \min\left\{\frac{\varepsilon x_0^2}{2}, \frac{x_0}{2}\right\}$.

Then

$$\left|\frac{1}{x} - \frac{1}{x_0}\right| \leq \frac{2}{x_0^2}|x - x_0| < \frac{\cancel{2}}{\cancel{x_0^2}} \cdot \frac{\cancel{x_0^2}}{\cancel{2}}\varepsilon = \varepsilon. \qquad \square$$

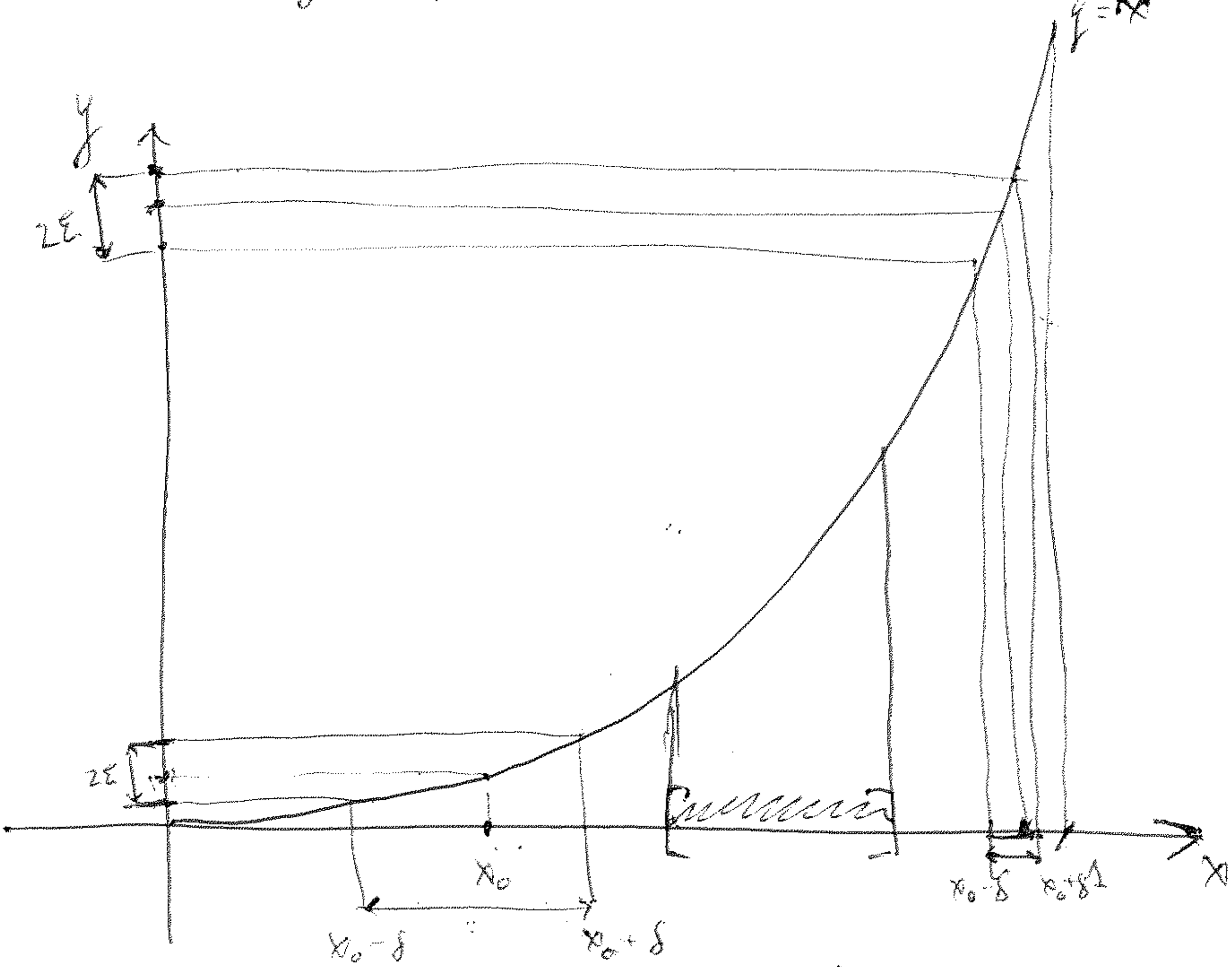
<u>Note.</u> δ depends on ε and x_0.

Uniform continuity. 10/04/14

Lec 17

Def. Suppose $f: I \to \mathbb{R}$, where I is an interval.

We say f is <u>uniformly continuous</u> on I if for every $\varepsilon > 0$, there exists $\delta > 0$ s.t. if $x, y \in I$ and $|x-y| < \delta$, then $|f(x) - f(y)| < \varepsilon$.



Example.

1) $f(x) = x$, on $\mathbb{R}$.

Uniformly contin.? YES

Take $\varepsilon > 0$. Take $\delta = \varepsilon$.

If $|x-y| < \delta$, then

$|f(x) - f(y)| = |x-y| < \delta = \varepsilon$.

---

2) $f(x) = x^2$ on $(0, \infty)$.

Unif. cont.? NO.

Assume it is.

Fix $\varepsilon > 0$. There is $\delta > 0$ s.t.

$x, y \in (0, \infty)$ and $|x-y| < \delta \Rightarrow |x^2 - y^2| < \varepsilon$.

Choose $y = x + \frac{\delta}{2}$. Then, if $x \in (0, \infty)$,

$y \in (0, \infty)$ and $|x-y| = \frac{\delta}{2} < \delta$.

Consider $|x^2 - y^2|$. We have

$$|x^2 - y^2| = \left|x^2 - \left(x + \frac{\delta}{2}\right)^2\right| = \left|\cancel{x^2} - \cancel{x^2} - x\delta - \frac{\delta^2}{4}\right|$$

$$= \left|-x\delta - \frac{\delta^2}{4}\right| = x\delta + \frac{\delta^2}{4} > x\delta.$$

Choose $x = \frac{\varepsilon}{\delta}$.

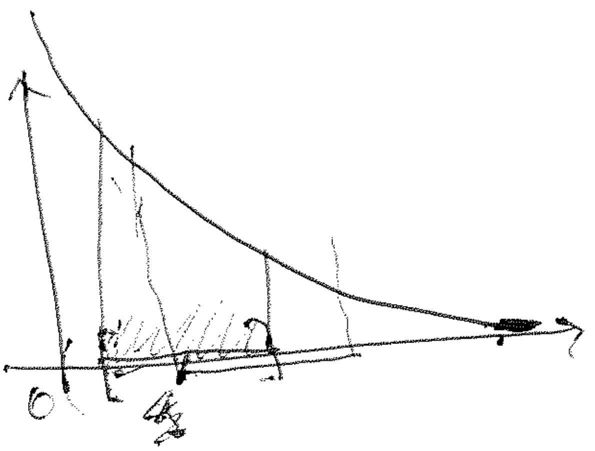
Then

$|x^2 - y^2| > x\delta = \varepsilon$.

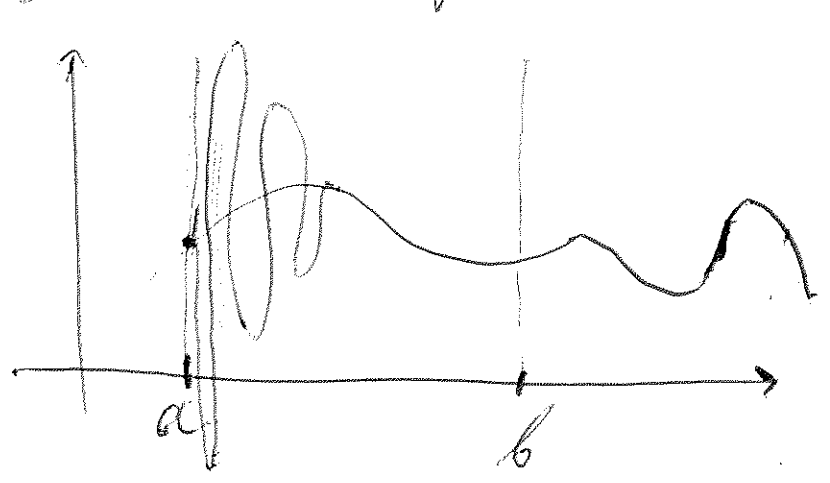
But this contradicts the definition.

$\square$

<u>Exercise.</u> $\frac{1}{x}$ unif. cont on $(0,1)$? NO.



<u>Theorem.</u> Suppose f is continuous on a closed, bounded interval $[a,b]$. Then f is uniformly cont. on $[a,b]$.



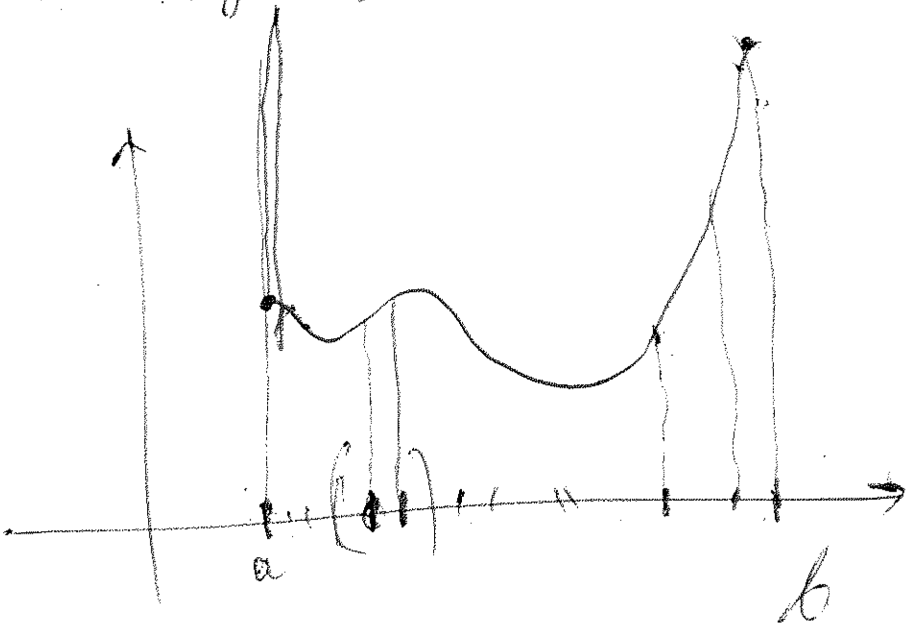
Proof. Want to prove f is unif cont.

For every $\varepsilon > 0$, there is $\delta > 0$ s.t. if $x, y \in [a,b]$ and $|x-y| < \delta$, then $|f(x) - f(y)| < \varepsilon$.

Proof by contradiction.

Assume unif. cont. fails.

~~Theare~~ There exists $\varepsilon_0 > 0$ such that for every $\delta > 0$, there exist $x, y \in [a,b]$ s.t. $|x-y| < \delta$ but $|f(x) - f(y)| \geq \varepsilon_0$.



$\delta = 1$

$\delta = \frac{1}{2}$

$\delta = \frac{1}{3}$

Take $\delta = 1$. There exist $x_1, y_1 \in [a,b]$ s.t. $|x_1 - y_1| < \delta = 1$ but $|f(x) - f(y)| \geq \varepsilon_0$.

There exist $x_2, y_2 \in [a,b]$ s.t. $|x_2 - y_2| < \frac{1}{2}$ but $|f(x_2) - f(y_2)| \geq \varepsilon_0$.

. . . -

For $\delta = \frac{1}{n}$, there are $x_n, y_n \in [a,b]$ s.t. $|x_n - y_n| < \frac{1}{n}$ but $|f(x_n) - f(y_n)| \geq \varepsilon_0$.

The sequence $(x_n)_{n=1}^{\infty}$ is bounded. Therefore it has a convergent subseq. $(x_{n_k})_{k=1}^{\infty}$.

Set $\lim\limits_{k \to \infty} x_{n_k} = x_0$.

<u>Claim.</u> $\lim\limits_{k \to \infty} y_{n_k} = x_0$. Why?

Indeed. $|y_{n_k} - x_0| = |(y_{n_k} - x_{n_k}) + (x_{n_k} - x_0)|$

$\leq \frac{1}{n_k} + |x_{n_k} - x_0|$.

Now, we are assuming f is contin. at x_0. But then

$$|f(x_{n_k}) - f(x_0)| < \frac{\varepsilon_0}{4} \text{ and}$$

$$|f(y_{n_k}) - f(x_0)| < \frac{\varepsilon_0}{4} \text{ for}$$

k large enough.

Then

$$|f(x_{n_k}) - f(y_{n_k})|$$

$$= |f(x_{n_k}) - f(x_0) + f(x_0) - f(y_{n_k})|$$

$$\leq |f(x_{n_k}) - f(x_0)| + |f(y_{n_k}) - f(x_0)|$$

$$< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon_0}{2}.$$

But we assumed

$$|f(x_n) - f(y_n)| \geq \varepsilon_0.$$

Contradiction. $\square$

Lec. 18

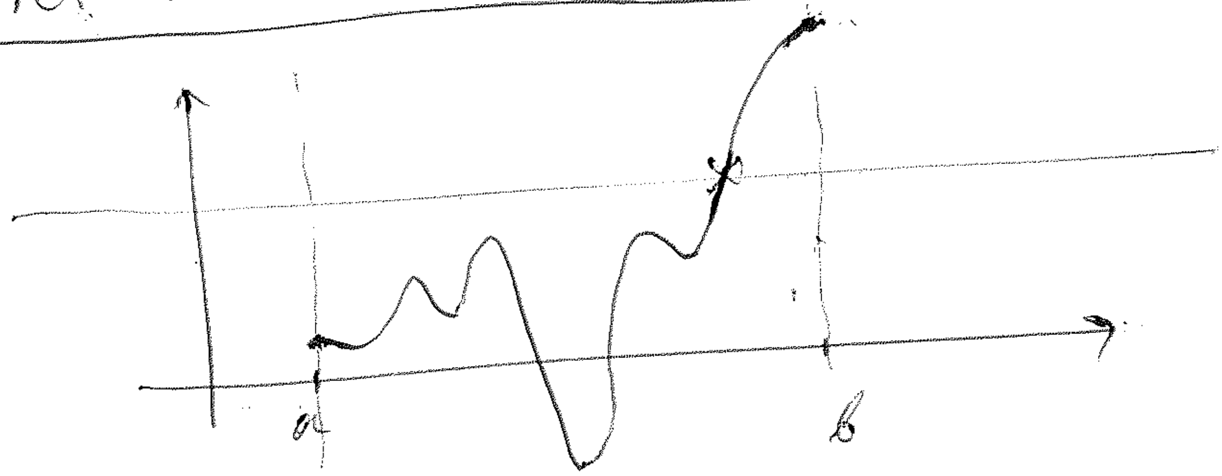
Remarks. 1) Uniform continuity on I implies unif. continuity on any subset of I.

2) The converse is false in general:

Ex. a) $f(x) = x$ is unif. cont. on $\mathbb{R}$.

b) $f(x) = x^2$ is unif. cont. on $(0, 10)$.

Intermediate value theorem.



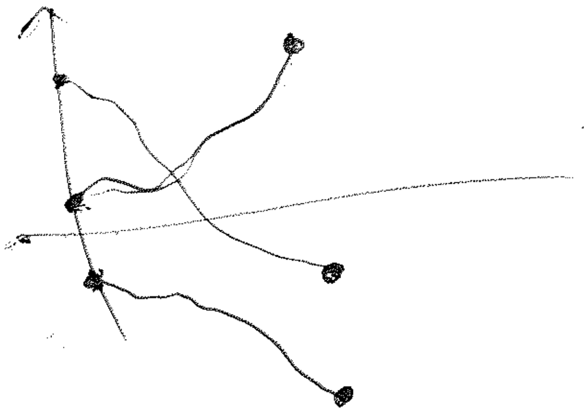
Theorem (I.V.T.).

Assume $f : [a, b] \to \mathbb{R}$. Take $\lambda \in \mathbb{R}$ s.t. $f(a) < \lambda < f(b)$ or $f(a) > \lambda > f(b)$. Then there exists $c \in (a, b)$ s.t. $f(c) = \lambda$.

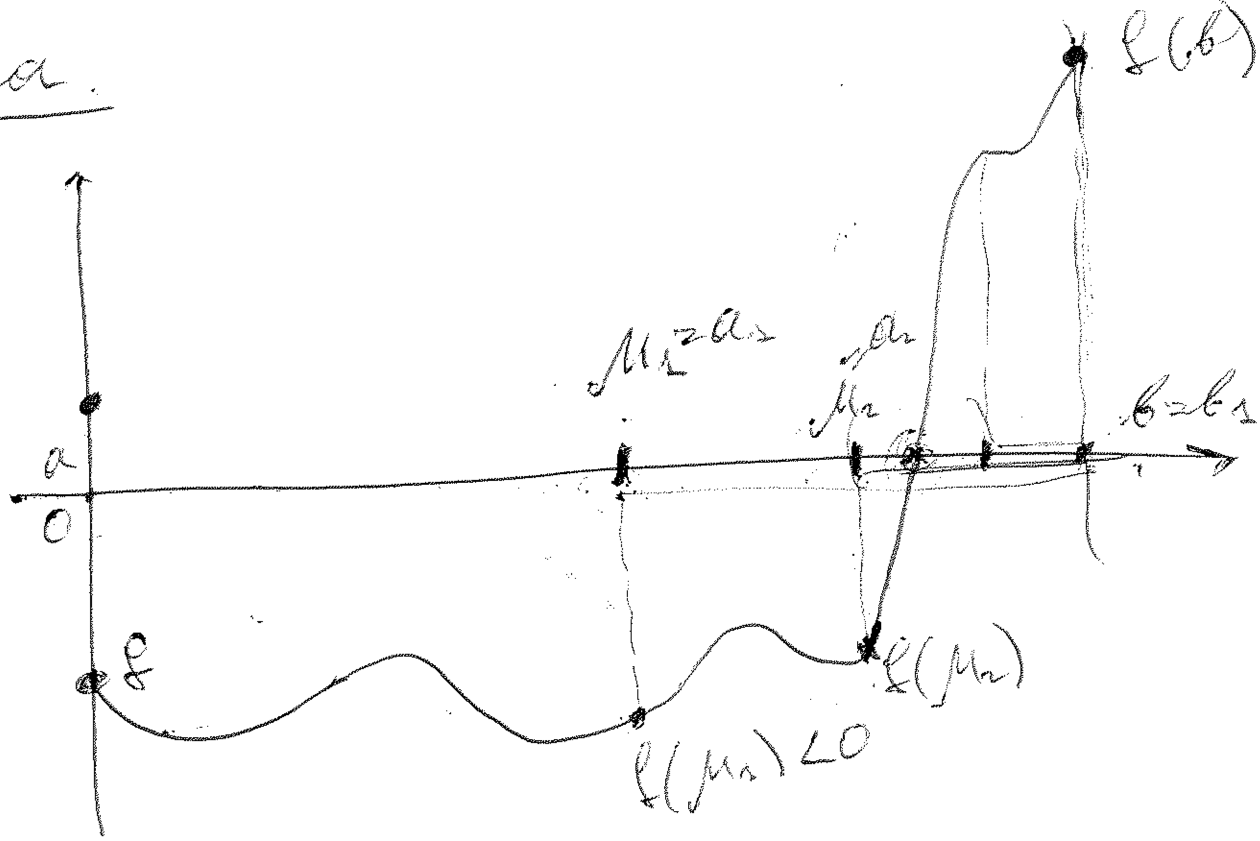
Proof.

Without loss of generality, assume $\lambda = 0$. (If $\lambda \neq 0$, simply consider $f(x) - \lambda$ instead of f).

Assume $f(a) < 0 < f(b)$.
(If not, simply consider $-f(x)$ instead of $f(x)$).



Idea.



Denote $\mu_1 = \frac{a+b}{2}$.

If $f(\mu_1) > 0$, set $a_1 = a$, $b_1 = \mu_1$.

If $f(\mu_1) < 0$, set $a_1 = \mu_1$, $b_1 = b$.

Denote $\mu_2 = \frac{a_1 + b_1}{2}$,

If $f(\mu_2) > 0$, set $a_2 = a_1$, $b_2 = \mu_2$.

$\boxed{\text{If } f(\mu_2) < 0}$

Otherwise, set $a_2 = \mu_2$, $b_2 = b_1$.

Continue this. Get a sequence $((a_n, b_n))_{n=1}^{\infty}$ such that

1. $(a_n)_{n=1}^{\infty}$ is nondecreasing,
   $(b_n)_{n=1}^{\infty}$ is nonincreasing.
2. $(a_n)_{n=1}^{\infty}$ is bounded by b,
   $(b_n)_{n=1}^{\infty}$ is bounded by a.
3. $b_n - a_n = \frac{b-a}{2^n}$.

There exist p_a and p_b s.t.

$\lim_{n\to\infty} a_n = p_a$, $\lim_{n\to\infty} b_n = p_b$.

Furthermore, $|b_n - a_n| = \frac{b-a}{2^n} \Rightarrow p_a = p_b \ (= p)$

f is continuous; therefore,

$$\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} f(b_n) = f(p).$$

We know $f(a_n) < 0$ and $f(b_n) > 0$. Squeeze $\Rightarrow$ ~~$f(a_n) = f(b_n) = 0$.~~

$$\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} f(b_n) = 0.$$

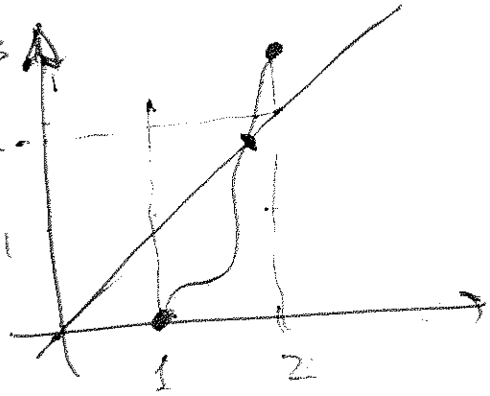
Thus, $f(p) = 0$. $\square$

Applications of I.V.T.

1) $f: [1,2] \longrightarrow \mathbb{R}$, continuous.

Assume $f(1) = 0$, $f(2) = 3$.

Prove: there is $x \in (1,2)$ s.t. $f(x) = x$.



Proof. Consider the function

$$g(x) = f(x) - x.$$

Then $g(1) = -1$, $g(2) = 1$.

IVT $\Rightarrow$ there is $x \in (1,2)$ s.t. $g(x) = f(x) - x = 0$. Then $f(x) = x$.